

Design of IIR filters for second-order derivative estimation

Petar Stančić, Matija Špeletić, Ivana Đorđević, Stevica Cvetković, and Goran Stančić

Abstract – A new method for design of digital IIR filters for input signal second-order derivative estimation is described in this paper. Proposed filters have parallel structure utilizing allpass IIR filters in both branches. Due to the specific configuration, only odd-order real filters can be obtained. Obtained differentiators demand minimum number of multipliers, compared to other approaches, since allpass filters of order N can be realized with only N multipliers. The proposed procedure can be generalized for design of partial differentiators.

Keywords - Differentiators, All-pass filters, Parallel configuration, Near-linear phase.

I. INTRODUCTION

The digital differentiator filters are used in various applications in signal processing, control systems, and communications. In control systems they are part of PID controllers, in signal processing differentiator can detect sudden spikes in time series data, heartbeat in ECG signal process, or edges in image processing. In inertial navigation systems it may be used to determine rates of change of acceleration. A variety of methods have been developed to derive a digital differentiator. Most of methods rely on some sort of an iterative procedure to find the coefficients of the transfer function. Some methods make use of metaheuristic algorithms like swarm intelligence, bat algorithm, or a genetic algorithm. Such a formulation of the design was proposed in [1], where differentiators are obtained by sequential partial optimization. It includes explicit equations for phase control and consequently provides lower phase error compared to other methods. In recent years, several papers have appeared on the design of low-pass and full-band differentiators using a novel parallel configuration [2], [3]. In this work, the proposed configuration is employed for the realization of a second-order differentiator.

II. FILTER CONFIGURATION

For realization of the IIR filter for estimation of the input signal second-order derivation we adopted a structure

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represented in Fig. 1.

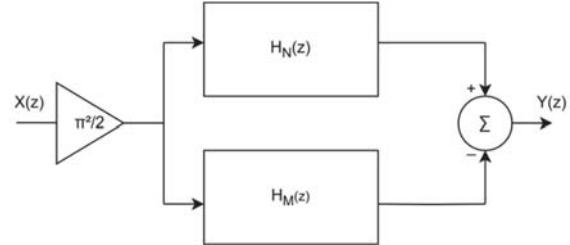


Fig. 1. Differentiator realized as parallel structure utilizing allpass filters $H_N(z)$ and $H_M(z)$

The transfer function of the second-order differentiator of order $N+M$ is given with

$$H(z) = \frac{Y(z)}{X(z)} = \pi^2 (H_M(z) - H_N(z)) / 2 \quad (1)$$

where $H_M(z)$ and $H_N(z)$ are allpass filters of order M and N , respectively.

$$H_M(e^{j\omega}) = |H_M(e^{j\omega})| e^{j\varphi_M(\omega)} = e^{j\varphi_M(\omega)} \quad (2)$$

$$H_N(e^{j\omega}) = |H_N(e^{j\omega})| e^{j\varphi_N(\omega)} = e^{j\varphi_N(\omega)} \quad (3)$$

The magnitude and phase responses of the differentiator are directly related to the phase characteristics of the employed all-pass filters and are given by the following expressions:

$$|H(e^{j\omega})| = \pi^2 \left| \sin \frac{\varphi_M(\omega) - \varphi_N(\omega)}{2} \right| \quad (3)$$

$$\varphi(\omega) = \frac{\pi + \varphi_M(\omega) + \varphi_N(\omega)}{2} \quad (4)$$

From expressions (3) and (4), it is straightforward to conclude that the design of the differentiator reduces to the problem of approximating the phases of the all-pass filters. This approach allows simultaneous control of both the magnitude and phase characteristics of the differentiator.

The ideal analog filter for second-order derivation has transfer function

$$H_{id}(s) = s^2 \quad (5)$$

from which it follows that the magnitude response depends on the square of the frequency, i.e.,

$$H_{id}(j\omega) = -\omega^2 \quad (6)$$

The same magnitude characteristic has corresponding digital differentiator from zero to Nyquist frequency. To achieve a such characteristic in ideal case, all-pass filters phases need to satisfy next expressions: In the ideal case, the

differentiator could have an ideal amplitude if the phases of the two all-pass filters differ by exactly $2\arcsin(\frac{\omega^2}{\pi^2})$. From equation (4), in order for the differentiator to also have a linear phase, the nonlinear term must be split into two equal parts with opposite signs. We thus conclude that, in the ideal case, the following holds:

$$\varphi_{Nid}(\omega) = -K\omega - \arcsin \frac{\omega^2}{\pi^2} \quad (7)$$

$$\varphi_{Mid}(\omega) = -K\omega + \arcsin \frac{\omega^2}{\pi^2} \quad (8)$$

The relation that the orders of the all-pass filters must satisfy ($N=M+1$) allows the determination of the slope of the ideal linear phase K based on equations (7) and (8). Starting from the condition satisfied by the phase of the all-pass filter,

$$\varphi_{Mid}(\pi) = -K\pi + \arcsin(1) = -\pi(K - 0.5) \quad (9)$$

the slope is obtained as:

$$K = N - 0.5 = M + 0.5 \quad (10)$$

It should be noted that the parallel all-pass structure allows achieving the desired amplitude with less hardware if a delay line is used instead of a IIR all-pass filter $H_N(z)$. However, the phase of such a differentiator will deviate from the ideal linear phase by a term of $\arcsin(\omega^2/\pi^2)$ (based on equation (4)), because in this case the phase of the IIR all-pass filter must differ from the ideal by a term of $2\arcsin(\omega^2/\pi^2)$.

III. INITIAL SOLUTION DETERMINATION

The all-pass filters transfer functions are

$$H_M(z) = \frac{b_M + b_{M-1}z^{-1} + \dots + z^{-M}}{1 + b_1z^{-1} + \dots + b_Mz^{-M}} \quad (11)$$

$$H_N(z) = \frac{a_N + a_{N-1}z^{-1} + \dots + z^{-N}}{1 + a_1z^{-1} + \dots + a_Nz^{-N}} \quad (12)$$

with corresponding phase responses

$$\varphi_M(\omega) = -M\omega + 2\arctan \frac{\sum_{i=1}^M b_i \sin(i\omega)}{1 + \sum_{i=1}^M b_i \cos(i\omega)} \quad (13)$$

$$\varphi_N(\omega) = -N\omega + 2\arctan \frac{\sum_{i=1}^N a_i \sin(i\omega)}{1 + \sum_{i=1}^N a_i \cos(i\omega)} \quad (14)$$

Most digital filter design procedures are based on iterative methods. This also applies to the design of a second-order differentiator using the proposed structure. A good initial solution allows convergence to the final solution while maintaining stability. The procedure aims to find an equiripple solution, which requires that the initial solution has the same number of extrema so that a sufficient number of equations can be formed at each step. It is desirable that the extrema of the initial solution have similar values, because in the first few iterations extrema that are much smaller than the remaining ones may be lost.

Thus, the phase error curve must have $N+1$ extrema, which means that N zeros ω_{zi} of the phase error function will be located between them. The initial solution for the all-pass filter $H_N(z)$ is obtained by solving a system of equations

$$\varphi_N(\omega_{zi}) + K\omega_{zi} + \arcsin \frac{\omega_{zi}^2}{\pi^2} = 0, i = 1, \dots, N \quad (15)$$

where

$$\omega_{zi} = \frac{\pi}{N+1}i, i = 1, 2, \dots, N \quad (16)$$

The other all-pass filter $H_M(z)$ is determined by solving

$$\varphi_M(\omega_{zi}) + K\omega_{zi} - \arcsin \frac{\omega_{zi}^2}{\pi^2} = 0, i = 1, \dots, M \quad (17)$$

IV. ALL-PASS FILTER DESIGN

In this step, the coefficients of the all-pass filter transfer function are determined so as to minimize the maximum phase error of the all-pass filter with respect to the ideal value, which ensures the desired magnitude and phase responses of the resulting differentiator using (13) and (14). The iterative procedure belongs to the class of minimax algorithms, since the goal is to obtain an equiripple phase error. In each iteration step, it is necessary to solve a system of equations

$$\varphi_N(\omega_i) - \varphi_{Nid}(\omega_i) = (-1)^i \varepsilon d_i, i = 1, 2, \dots, N+1 \quad (18)$$

where ω_i are frequencies at which extrema of the phase error function are located. Parameters d_i offer additional control of obtained error. For all values equal to unity obtained phase error will be equiripple. It is well known that spectrum of input signal has more significant components at low frequencies. For that reason one can expect better time domain behaviour of differentiator if approximation error is lower in lower band. We design all-pass filters up to the order $N=20$. The obtained equiripple solutions were starting points to quasi-equiripple design. Numerous experiments have shown that most improvements can be achieved if one allow the last extremum to be bigger than others (all others are the same, i.e., N first parameters d_i have value equal to one, while $d_{N+1} > 1$).

III. OBTAINED RESULTS

For the evaluation of the obtained differentiators, the following parameters are used: the maximum magnitude and phase errors, denoted by δ_M and δ_P , the magnitude mean relative error (MRE), and the mean phase error (MPE). These values are listed in Table I. The table also lists the magnitude of the pole closest to the unit circle r_i , i.e., to the stability boundary. For all obtained differentiators, nevertheless the error is equiripple or quasi-equiripple, maximum pole module does not exceed the value of 0.6. In general, IIR filter of order K demands

$2K+1$ multiplications. The proposed differentiator, thanks to parallel all-pass configuration, demands only $K+1$ multiplications.

TABLE I
Evaluation metrics for differentiators of various orders

order	δ_M	$\delta_P [^\circ]$	MRE	MPE $[^\circ]$	max r_i
3	0.8342	13.1475	0.4387	7.7054	0.4912
5	0.6457	10.1380	0.3691	5.9463	0.5869

Obtained magnitude, along with corresponding absolute and relative error, and phase response including appropriate error are shown in Fig. 2 and Fig. 3, respectively.

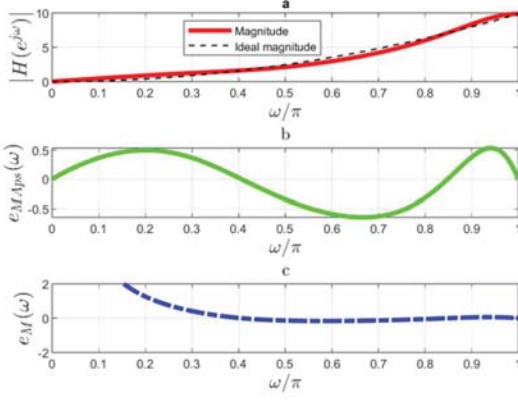


Fig. 2. Fifth-order differentiator: (a) magnitude response, (b) absolute, and (c) relative error

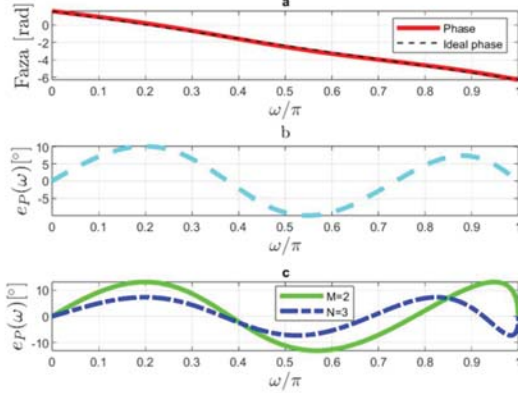


Fig. 3. Fifth-order differentiator: (a) Phase response, phase errors of (b) differentiator, and (c) all-pass filters

The similar results are obtained for the third order differentiator. From Fig. 2 it can be observed that the absolute error is nearly uniform over the entire frequency range from zero to the Nyquist frequency, whereas the relative error is increased at very low frequencies. It should be emphasized that the errors of the differentiator are not equiripple; only the phase errors of the all-pass filters exhibit equiripple behavior. Owing to the typical property of input signals having spectra richer at low frequencies,

the design procedure is modified to reduce the relative error in the vicinity of zero frequency at the expense of increased error at higher frequencies. The design still focuses solely on the all-pass filters, aiming to improve the characteristics of the resulting differentiator. Extensive experiments have shown that it is sufficient to increase only the value of the last extremum of the phase error, with the best results obtained for values of $d_{N+1}(d_{M+1})$ in the vicinity of 6.

TABLE II
Performance metrics for differentiators with non-equiripple errors

order	δ_M	$\delta_P [^\circ]$	MRE	MPE $[^\circ]$	max r_i
3	1.3858	8.2642	0.4907	3.7323	0.3621
5	0.6917	6.8176	0.2332	3.8229	0.5423

By comparing the data presented in Tables 1 and 2, an increase in the maximum magnitude error can be observed; however, this maximum occurs at higher frequencies. At the same time, the error at low frequencies is reduced, along with a significant reduction in phase error. Furthermore, both the MRE and MPE parameters are reduced by approximately 40%.

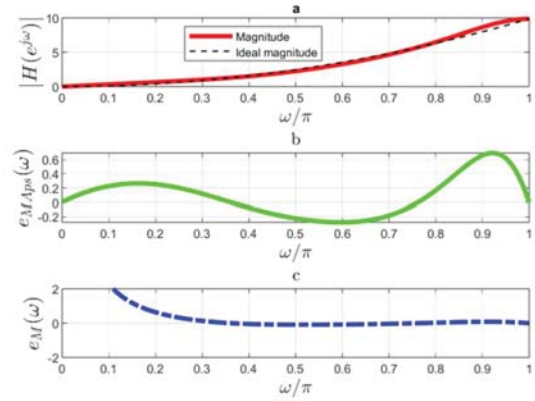


Fig. 4. Fifth-order differentiator: (a) magnitude response, (b) absolute, and (c) relative error

The influence of the parameter d_{N+1} on the magnitude relative error and phase error is shown in Fig. 5. It is noticeable that increasing d_{N+1} has a beneficial effect on both the magnitude and phase characteristics at low frequencies, which was the intended objective. Since the magnitude of the differentiator depends on the phase differences of the all-pass filters, the minimum error is achieved when the phase errors of the all-pass filters overlap (recall that they approximate different ideal phase functions). In the first step, a higher-order all-pass filter of order N is designed to exhibit an equiripple or quasi-equiripple phase error, and then the second all-pass filter is designed so that the frequencies and values of the first M extrema match those of the first filter. Obtained results for 5th order differentiator are presented in Fig. 6 for $d_{N+1} = 1$ and $d_{N+1} = 6$.

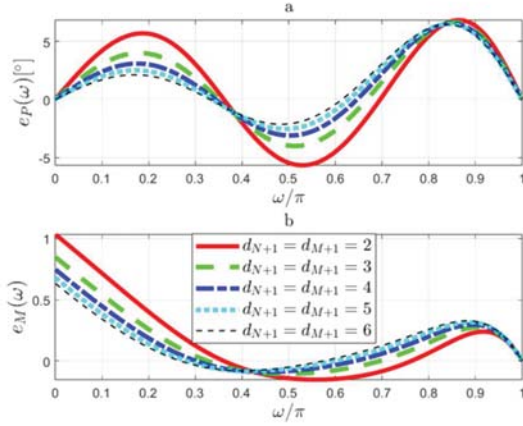


Fig. 5. Influence of the parameter d_{N+1} on (a) the differentiator phase and (b) magnitude relative errors

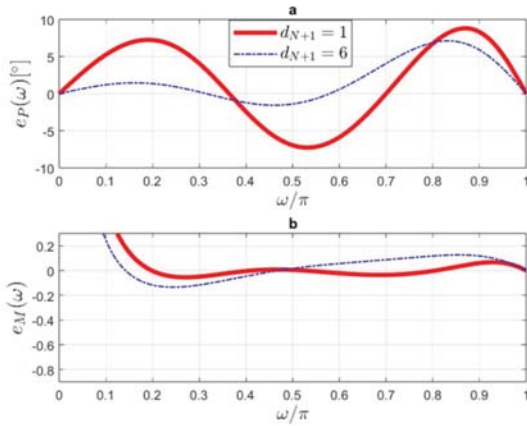


Fig. 6. (a) The phase error and (b) relative magnitude error of 5th order differentiator for two values of parameter d_{N+1}

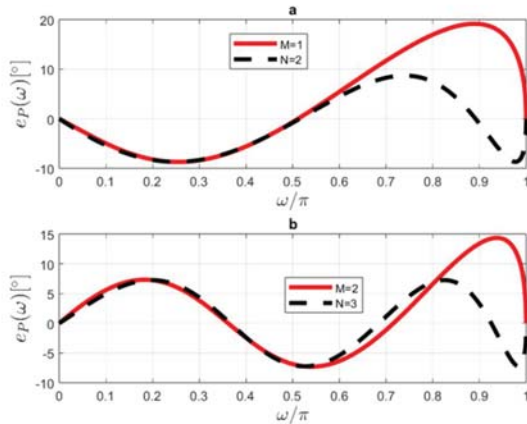


Fig. 7. The all-pass filters phase errors for (a) 3rd and (b) 5th order differentiator for $d_{N+1} = 1$

The all-pass filters phase errors of differentiators of order 3 and 5 are shown in Fig. 7. As explained earlier, the

first M extrema of the all-pass filters phase error are exactly the same and positioned at the same frequencies. The proposed fifth-order differentiator is realized using six multipliers. A standard IIR filter requiring a similar number of multiplications would be of order 2 (five multipliers) or 3 (seven multipliers). In the available literature, no works were found addressing the design of second-order differentiators. In further research, we will compare the obtained magnitude and phase errors for filters of similar or identical (if possible) complexity, where one solution is the double application of a first-order differentiator (i.e., squaring the transfer function), and the other solution is the proposed second-order differentiator.

IV. CONCLUSION

A new method for design of IIR filters for second-order derivative estimation is described in this paper. Given procedure can be generalized for design of partial differentiators. Due to specific configuration, only odd-order real filters can be obtained. Obtained differentiators demand minimum number of multipliers for given order, compared to other approaches. The subject of further research will be the modification of the described procedure so that it can be applied to the design of fractional differentiators. A comparison of the results obtained by the double application of a first-order differentiator (which computes the first derivative of the input signal) and the proposed second-order differentiator (which directly computes the second derivative of the input signal) indicates that, when the signal corresponding to the first derivative is not of interest, a filter of the same complexity yields a smaller magnitude error and approximately the same phase error.

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